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Exact linear invariants and quantum effects in the early universe

I.A. Pedrosa, Claudio Furtado, Alexandre Rosas *

Departamento de Física, CCEN, Universidade Federal da Paraíba, Caixa Postal 5008, 58059-900 João Pessoa, Brazil

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Abstract

In this contribution we use linear invariants and the dynamical invariant method in the framework of the linear Schrödinger equation to study scalar fields in a Friedman–Robertson–Walker spacetime, obtaining exact wave functions to this problem. In addition, we construct Gaussian wave packet solutions and calculate the quantum fluctuations as well as the quantum correlations for each mode of the quantized scalar field.

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1. Introduction

The harmonic oscillator model is one of the most important models in classical and quantum physics. It is exactly solvable quantum mechanically and offers a wide range of applications in the description of physical systems in areas such as quantum optics [1,2] and gravitation [3]. As a non-stationary system, a time-dependent quantum oscillator can also be exactly solved and the dynamical invariant method, devised by Lewis and Riesenfeld [4], gives a typical and powerful method to study this system [5–8].

Another problem that has attracted great interest for its fundamental physical perspective is that of a scalar field placed in a Friedman–Robertson–Walker (FRW) spacetime [9–12,14]. The behavior of matter scalar quantum fields as well as gravitational waves [12] is governed by the Einstein equations of the time-dependent type that can be mapped in an equation of the time-dependent oscillator. Thus the problem of both particle creation in a metric field fluctuation during a cosmological evolution is reduced to solve the quantum time-dependent harmonic oscillator. This problem can be treated as a time-dependent harmonic oscillator since a massive scalar field when appropriately decomposed into modes, inherits a time dependence from a time-dependent spacetime background [15]. Recently, some of us (Pedrosa and Furtado) [16] have studied the problem of a scalar field in FRW spacetime using the invariant method together with an exact time dependent quadratic invariant.

The connection of cosmology with some process in quantum optics has received some attention. The idea of adoption of the language of the squeezed states to cosmological particle creation was first introduced by Grisshchuk and Sidorov [12,13]. In this context, a coherent state representation for a scalar field minimally coupled to a gravitational field was constructed [9] and the language of quantum optics was used to analyze the existence of squeezed states in a cosmological model [17]. Matacz [18] using squeezed state formalism derived the coherent state representation of quantum fluctuations in a expanding universe. Further, Hu et al. [17] have applied squeezed states formalism to discuss the role of initial states in particle creation and have pointed out that squeeze and rotation operator were first derived by Parker [19] in his analyzes of cosmological particle creation, based on the work by Kamefuchi and Umezawa [20]. Also, Matacs [18] have considered a squeezed vacuum of harmonic oscillator system with time-dependent frequency to study the coherent representation of quantum fluctuation in expanding universe.

In this contribution, instead of quadratic invariants, we use exact *linear invariants* to solve the linear Schrödinger equation for a scalar quantum field placed in a FWR spacetime. The

* Corresponding author.
E-mail addresses: iapedrosa@fisica.ufpb.br (I.A. Pedrosa), furtado@fisica.ufpb.br (C. Furtado), arosas@fisica.ufpb.br (A. Rosas).

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derivation of our solution is not only less lengthy but also more direct than that of Ref. [16], and it allows us to get the physical properties of the scalar quantum field more readily, as seen below. In addition, we construct Gaussian wave packet solutions and calculate the quantum fluctuations and correlations for each mode of the quantized field.

The remaining of this Letter is organized as follows. In Section 2 we derive the Hamiltonian density for a scalar field placed in a FRW spacetime. Next, in Section 3, we use exact linear invariants and the invariant method to solve the Schrödinger equation for the problem. Section 4 describes the construction of Gaussian wave packets solutions and shows how one can calculate the quantum fluctuations and correlations for each mode of the quantized field. Finally, we briefly summarize our results in Section 5.

2. Hamiltonian density of scalar fields in a FRW spacetime

Let us consider a FRW line element given by

$$ds^2 = -dt^2 + a^2(t) g_{ij}\, dx^i\, dx^j, \tag{1}$$

where g_{ij} is the metric of the three-dimensional flat spacetime and $a(t)$ is the universal scale factor of the universe. We further choose a real field $\Phi(\mathbf{x}, t)$, whose Lagrangian density is written as

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu}\, \partial_\mu \Phi\, \partial_\nu \Phi - \frac{1}{2}\left(\xi R + m^2\right)\Phi^2, \tag{2}$$

where m is the mass of the field quanta, ξR is the coupling between the scalar field and the gravitational field, ξ being a numerical factor and R the Ricci scalar. What is more, the scalar field may be decomposed on a complete basis $u_k(\mathbf{x}, t)$,

$$\Phi(\mathbf{x}, t) = \sum_k \left(a_k u_k + a_k^* u_k^*\right), \tag{3}$$

where

$$u_k(\mathbf{x}, t) = \frac{1}{\sqrt{V}} \exp(i\mathbf{k} \cdot \mathbf{x}) \frac{\phi_k^1 + i\phi_k^2}{\sqrt{2}}, \tag{4}$$

is normalized in the volume V. In Eq. (4) we have split ϕ_k into its real and imaginary parts. Hence, using Eqs. (2) to (4), we obtain the action

$$S = \frac{1}{2} \sum_k \sum_{j=1,2} \int a^3\, dt \left(\dot{\phi}_k^j - \omega_k^2 \phi_k^{j^2}\right), \tag{5}$$

where the "angular frequency" is given by

$$\omega_k^2(t) = \frac{k^2}{a^2(t)} + m^2 + \xi R, \tag{6}$$

and the dot denotes a time derivative. Eq. (5) shows that the modes $\mathbf{k}$ and j are independent and may be considered separately. Consequently, each mode contributes to the Hamiltonian with

$$\mathcal{H}_k^j = \frac{1}{2a^3}\left(\Pi^{j^2} + a^6 \omega_k^2 \phi_k^{j^2}\right). \tag{7}$$

Therefore, the classical equations of motion for the fields ϕ_k^j

$$\ddot{\phi}_k^j + 3\frac{\dot{a}}{a}\dot{\phi}_k^j + \omega_k^2 \phi_k^j = 0, \tag{8}$$

follows from this Hamiltonian density. Furthermore, canonical quantization ($\Pi_k^j \to -i\hbar\partial/\partial\phi_k^j$) leads to the Hamiltonian operator

$$\hat{\mathcal{H}}_k^j = \frac{1}{2a^3}\left(-\hbar^2 \frac{\partial^2}{\partial \phi_k^{j^2}} + a^6 \omega_k^2 \phi_k^{j^2}\right), \tag{9}$$

which of course depends explicitly on time. It is worth noticing that Eq. (8) is the equation of motion for the classical harmonic oscillator with time-dependent mass and frequency which may be obtained from the Hamiltonian [5]

$$H(t) = \frac{p^2}{2m(t)} + \frac{1}{2} m(t)\omega^2(t) q^2, \tag{10}$$

for each mode k_i ($i = 1, 2, 3$), with mass $m(t) = a^3(t)$ and frequency $\omega(t)$ defined as in Eq. (6). Thus, our problem can be mapped into an harmonic oscillator with time-dependent mass and frequency. This system has been studied with the aid of the dynamical invariant method [5–8].

3. Linear invariants and scalar fields in a FRW spacetime

For the system described by the Hamiltonian (7), the Schrödinger equation is

$$i\hbar \frac{\partial \Psi(\phi_k^j, t)}{\partial t} = \mathcal{H}_k^j \Psi\left(\phi_k^j, t\right). \tag{11}$$

According to the dynamical invariant method [4], the solutions of the Schrödinger equation (11) are related to the eigenfunctions of a Hermitian operator $I_k^j(t)$ which satisfies the following equation

$$\frac{dI_k^j(t)}{dt} = \frac{1}{i\hbar}\left[I_k^j(t), \mathcal{H}_k^j\right] + \frac{\partial I_k^j(t)}{\partial t} = 0. \tag{12}$$

The above condition allows one to write the solutions of the Schrödinger equation as

$$\Psi_\lambda\left(\phi_k^j, t\right) = e^{i\mu_\lambda(t)} \varphi_\lambda\left(\phi_k^j, t\right), \tag{13}$$

where $\varphi_\lambda(\phi_k^j, t)$ is an eigenfunction of $I_k^j(t)$ with time-independent eigenvalues λ and $\mu_\lambda(t)$ is a time-dependent phase function satisfying

$$\hbar \frac{d\mu_\lambda(t)}{dt} = \langle \varphi_\lambda | i\hbar \frac{\partial}{\partial t} - \mathcal{H}_k^j | \varphi_\lambda \rangle. \tag{14}$$

Quadratic invariant operators satisfying Eq. (12) are innumerable [4], but, as already commented, we are interested in dealing with a linear Hermitian invariant of the form

$$I_k^j(t) = \alpha_k(t)\phi_k^j + \beta_k(t)\Pi_k^j + \gamma_k(t), \tag{15}$$

where $\alpha_k(t)$, $\beta_k(t)$ and γ_k are time-dependent real functions to be determined. Since $I_k^j(t)$ must satisfy Eq. (12), these functions are related as

$$\dot{\alpha}(t) = a^3 \omega_k^2(t) \beta_k(t), \tag{16}$$

$$\dot{\beta}(t) = -\frac{\alpha_k(t)}{a^3}, \tag{17}$$

$$\dot{\gamma}(t) = 0. \tag{18}$$

Hence, from Eqs. (16) and (17) we find that

$$\ddot{\beta}_k(t) + 3\frac{\dot{a}}{a}\dot{\beta}_k(t) + \omega_k^2(t)\beta_k(t) = 0. \tag{19}$$

Once $\beta_k(t)$ is known, $\alpha_k(t)$ can be directly obtained from Eq. (16). Therefore, the linear invariant can be written as

$$I_k^j(t) = \beta_k(t)\Pi_k^j - a^3\dot{\beta}_k(t)\phi_k^j, \tag{20}$$

where, without loss of generality, we have set $\gamma(t) = \text{const} = 0$. Moreover, the eigenstates $|\varphi_\lambda\rangle$ of $I_k^j(t)$ form a continuous complete set whose time-independent eigenvalues λ are solution of equation [2,7,21]

$$I_k^j(t)\varphi_\lambda(\phi_k^j, t) = \lambda\varphi_\lambda(\phi_k^j, t), \tag{21}$$

with

$$\langle\varphi_\lambda(\phi_k^j, t)|\varphi_{\lambda'}(\phi_k^j, t)\rangle = \delta(\lambda - \lambda'). \tag{22}$$

It is straightforward to show that the eigenstates of $I_k^j(t)$ are

$$\varphi_\lambda(\phi_k^j, t) = \sqrt{\frac{1}{2\pi\hbar\beta_k(t)}} \times \exp\left[\frac{i}{\hbar\beta_k(t)}\left(\frac{a^3\dot{\beta}_k(t)}{2}\phi_k^{j^2} + \lambda\phi_k^j\right)\right]. \tag{23}$$

On the other hand, after the evaluation of the matrix element of the right-hand side of Eq. (14), the phase function may be written as

$$\mu_\lambda(t) = -\frac{\lambda^2}{2\hbar}\int_0^t \frac{1}{a^3(t')\beta_k^2(t')}dt'. \tag{24}$$

Therefore, the wave functions are given by

$$\Psi_\lambda(\phi_k^j, t) = \sqrt{\frac{1}{2\pi\hbar\beta_k(t)}} \times \exp\left[i\mu_\lambda(t) + \frac{ia^3\dot{\beta}_k(t)}{2\hbar\beta_k(t)}\phi_k^{j^2} + \frac{i\lambda\phi_k^j}{\hbar\beta_k(t)}\right], \tag{25}$$

and a general state is described by

$$\Psi(\phi_k^j, t) = \int_{-\infty}^{\infty} g(\lambda)\Psi_\lambda(\phi_k^j, t)\,d\lambda, \tag{26}$$

where $g(\lambda)$ determine the state. It is worth mentioning that when $\beta_k(t)$ vanishes the phase function $\mu_\lambda(t)$ diverges. In spite of this divergence, the wave functions (25) are always finite [2,21].

4. Gaussian wave packets, quantum fluctuations and quantum correlations

Gaussian wave packets are very common. Therefore, we turn our attention to this particular state. It is characterized by

$$g(\lambda) = \frac{\sqrt{b}}{(2\pi)^{1/4}}e^{-\frac{b^2}{4}\lambda^2}, \tag{27}$$

where b is a positive real constant. Thus, inserting Eqs. (25) and (27) into Eq. (26) and performing the integration we are left with

$$\Psi(\phi_k^j, t) = \left(\frac{2}{\pi}\right)^{1/4}\frac{\exp(-\frac{ia^3\dot{\beta}_k(t)}{2\hbar\beta_k(t)}\phi_k^{j^2})}{\sqrt{\hbar b\beta_k(t)}(1 + \frac{2if_k(t)}{\hbar b^2})} \times \exp\left[-\frac{\phi_k^{j^2}}{\hbar^2 b^2\beta_k^2(t)(1 + \frac{2if_k(t)}{\hbar b^2})}\right], \tag{28}$$

where

$$f_k(t) = \int_0^t \frac{1}{a^3(t')\beta_k^2(t')}dt'. \tag{29}$$

Moreover, the time-dependent probability density associated with the Gaussian wave packet is Gaussian for any time

$$\rho_k(\phi_k^j, t) = |\Psi(\phi_k^j, t)|^2 = \frac{1}{\sqrt{\pi}\sigma_k(t)}e^{-\phi_k^{j^2}/\sigma_k^2(t)}, \tag{30}$$

where

$$\sigma_k(t) = \sqrt{\frac{\hbar^2 b^2\beta_k^2(t)}{2}\left[1 + \frac{4f_k^2(t)}{\hbar^2 b^4}\right]} \tag{31}$$

is a time dependent width. Consequently, the center of the wave packet remains at $\phi_k^j = 0$ while its width change in time, as expected for oscillatory behavior [22–24]. Furthermore, it is readily verified that the wave function (28) is normalized and time-dependent probability density is conserved, i.e.,

$$\int_{-\infty}^{\infty} |\Psi(\phi_k^j, t)|^2\,d\phi_k^j = 1. \tag{32}$$

The quantum fluctuation for each mode of the quantized field in the state $\Psi(\phi_k^j, t)$ is also obtained after a straightforward calculation as

$$\Delta\phi_k^j = \sqrt{\langle\phi_k^{j^2}\rangle - \langle\phi_k^j\rangle^2} = \frac{1}{2\sqrt{U_k(t)}}, \tag{33}$$

$$\Delta\Pi_k^j = \sqrt{\langle\Pi_k^{j^2}\rangle - \langle\Pi_k^j\rangle^2} = \frac{\hbar\sqrt{U_k^2(t) + V_k^2(t)}}{\sqrt{U_k(t)}}, \tag{34}$$

where $U_k(t)$ and $V_k(t)$ are given by

$$U_k(t) = \frac{1}{\hbar^2 b^2\beta_k^2(t)(1 + \frac{4f_k^2(t)}{\hbar^2 b^4})}, \tag{35}$$

$$V_k(t) = \frac{a^3\dot{\beta}_k(t)}{2\hbar\beta_k(t)} + \frac{2f_k(t)}{\hbar^3 b^4\beta_k^2(t)(1 + \frac{4f_k^2(t)}{\hbar^2 b^4})}. \tag{36}$$

Hence, the uncertainty product takes the form

$$\left(\Delta\phi_k^j\right)\left(\Delta\Pi_k^j\right) = \frac{\hbar}{2}\sqrt{1 + \left(\frac{V_k(t)}{U_k(t)}\right)^2}. \tag{37}$$

If we require that the minimum uncertainty is $\hbar/2$ for a given time τ, then we must have $V_k(\tau) = 0$, or

$$\dot{\beta}_k(\tau) = \frac{4 f_k(\tau)}{a^3\beta_k(\tau)(\hbar^2 b^4 + 4 f_k^2(\tau))}. \tag{38}$$

Therefore, if we start with a minimum uncertainty packet, that is, $(\Delta\phi_k^j)(\Delta\Pi_k^j) = \hbar/2$ for $\tau = 0$, then the above condition is obviously reduced to $\dot{\beta}_k(0) = 0$ (note that, by definition, $f_k(0) = 0$). Further, $\beta_k(0)$ is related to the initial width of the Gaussian wave packet (see Eq. (30)). Hence, the initial conditions needed to solve Eq. (19) are completely set.

Finally, we calculate the quantum correlations for the quantized field modes. They are defined as [25]

$$C_{1,1} = \frac{1}{2}\left\langle\left\{\left(\phi_k^j - \langle\phi_k^j\rangle\right), \left(\Pi_k^j - \langle\Pi_k^j\rangle\right)\right\}\right\rangle \tag{39}$$

where anti-commutator is represented by $\{,\}$. Thence, using Eqs. (28) and (39), the quantum correlations are found to be

$$C_{1,1} = \frac{\hbar V_k(t)}{2U_k(t)}. \tag{40}$$

Therefore, even when the initial state is uncorrelated, quantum correlations develops as time goes on. What is more, the appearance of the correlations comes with an increase in the uncertainty. In fact, the condition (38) implies that the correlation vanishes when the uncertainty is minimum. Actually, the uncertainty and the quantum correlations are directly related as

$$\left(\Delta\phi_k^j\right)\left(\Delta\Pi_k^j\right) = \frac{\hbar}{2}\sqrt{1 + \left(\frac{2}{\hbar}C_{1,1}\right)^2}. \tag{41}$$

We end this section noticing that the absence of correlation in the uncertainty product minimum is expected because any correlation would be a constraint on the minimization of the uncertainty product and hence it would not reach its minimum value [26].

5. Summary

In this contribution we have used linear invariants and the dynamical invariant method to investigate solutions for the problem of a scalar quantum field placed in a FRW spacetime. Starting with the metric (1) and the Lagrangian (2), we obtained the time-dependent state of the system as a function of the solution of a second order ordinary differential equation (19). Furthermore, this later equation is completly determined by the scale factor of the metric $a(t)$ and the coupling between the scalar and the gravitational fields. Similarly, we have constructed Gaussian wave packets solutions whose probability density, quantum fluctuations, quantum correlations, and uncertainty product were determined also as function of solutions of Eq. (19). Finally, we would like to point out that the approach developted in this Letter can be useful to study other problems envolving creation of particles in a specific spacetime background [27–30].

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